ELASTIC-PLASTIC EQUILIBRIUM OF AN ACUTE-ANGLED WEDGE UNDER ULTIMATE RESISTANCE TO SHEAR AVERAGE TENSILE STRESS, AND RUPTURE

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The elastic-plastic plane strain state of an infinite wedge loaded by a uniform pressure on one face was examined in [1, 2], etc, where the state of an incompressible ideally plastic material corresponds to the condition of resistance to the maximal tangential stress

$$\tau_{\max} = (1/2)\,|\sigma_1 - \sigma_2| \leqslant k, \quad k = \text{const} > 0. \tag{1}$$

In the plane σ_1, σ_2 of the principal stresses (1) is interpreted by the strip $A_1A_2B_1B_2$ (Fig. 1).

Besides the limit of resistance to shear, the limit of resistance to rupture (to the maximal tensile stress) [5, 6] is taken into account in [3, 4]. Such an approach permits bounding the domain of computed normal stresses from above by the broken line A_1AMBB_1 since (1) is supplemented by the condition of resistance to rupture

$$\sigma_{\max} = \sigma_i \leqslant d\ (i = 1, 2),\ d = \text{const} > 0. \tag{2}$$

A description of the state of a plastic medium for the limit resistance to the average tensile stress which corresponds to the condition

$$\sigma = (\sigma_1 + \sigma_2)/2 \leqslant s,\ s = \text{const} > 0. \tag{3}$$

is given in [7, 8].

Starting from the conditions of resistance to shear (1) and to rupture (2), the domain of allowable stresses A_1AMBB_1 can be minimized for fixed points A and B by considering (1) in combination with (3) for $s = d - k$ and by conserving the condition of non-concavity of the domain A_1ABB_1 [6]. The domain in which the maximal stress components did not reach the limit values will be considered elastic.

The present paper is devoted to an analysis of the limit state of an elastic-plastic wedge in the stress domain (1) and (3), i.e., under ultimate resistance to shear and to the average tensile stress. Three characteristic values of the ultimate load q_1, q_2, q_3 are determined. The wedge is completely in the elastic state in the range $0 < q < q_1$. A zone of ultimate resistance to (2) or (3) occurs for $q_1 \leqslant q < q_2$ according to the model taken. Moreover, condition (1) is realized in the zone of ultimate resistance to shear in the range $q_2 \leqslant q < q_3$. For $q = q_3$ degeneration of the elastic zone into a line of stress discontinuity occurs. The load q_3 governs the ultimate carrying capacity of the elastic-plastic wedge. The solutions are compared for conditions (1)-(3).

In the general case the wedge is divided into three zones α, β, δ (Fig. 2). We construct a solution for a uniform loading by a load q along the face OA by starting from the presence of the zones α and β with rectilinear boundaries AOB and DOC and homogeneous stress states. We denote the stress components in these zones by the superscripts minus and plus. A homogeneous stress state of ultimate resistance to shear

$$\sigma_1^- = 0, \quad \sigma_2^- = -2k, \quad \tau_{\max}^- = (\sigma_1^- - \sigma_2^-)/2 = k. \tag{4}$$

is realized in the zone α subjected to compression. The condition of ultimate resistance to the average tensile stress

$$\sigma^+ = (\sigma_1^+ + \sigma_2^+)/2 = s, \quad \sigma_2^+ = -q, \quad \sigma_1^+ = 2s + q. \tag{5}$$

is realized in the zone β subjected to tension in one of the principal directions.

The points α and β in Fig. 1 correspond to the states (4) and (5) in the plane of the principal directions, the elements α_1, α_2 correspond to them in Fig. 2, and in Fig. 3 the stress states (4) and (5) are interpreted by the Moor diagrams of α and β.

Cheboksary. Translated from Zhurnal Prikladnoi Mekhaniki i Tekhnicheskoi Fiziki, No. 4, pp. 157-162, July-August, 1988. Original article submitted May 6, 1987.

 0021-8944/88/2904-0598$12.50

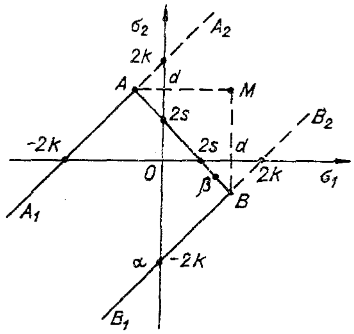

Fig. 1

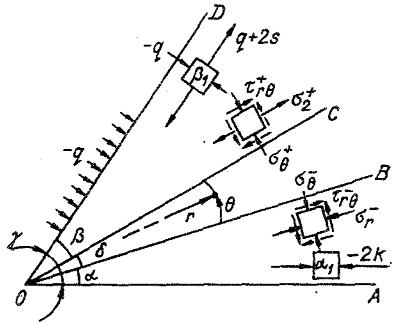

Fig. 2

Let us introduce the polar r, θ coordinate system with pole at the wedge apex O. The stress components in this coordinate system are determined in the zone α for r directed along OB by the relationships

$$\sigma_r^-,\ \sigma_\theta^- = -k(1 \pm \cos 2\alpha),\quad \tau_{r\theta}^- = k\sin 2\alpha. \tag{6}$$

For r directed along the boundary OC in the zone β, the stress components in the polar coordinate system are found from the relationships

$$\sigma_r^+,\ \sigma_\theta^+ = s \pm \rho\cos 2\beta,\quad \tau_{r\theta}^+ = \rho\sin 2\beta,\quad \rho = s + q. \tag{7}$$

In the zone δ we have a problem about an infinite elastic wedge BOC loaded along the faces OB and OC by pressures with components (6) and (7), respectively. Following [9, 1], we write the stresses in the elastic zone in the form

$$\sigma_r(\theta),\quad \sigma_\theta(\theta) = A - 2D\theta \pm (B\sin 2\theta + C\cos 2\theta),\quad \tau_{r\theta}(\theta) = D + B\cos 2\theta - C\sin 2\theta. \tag{8}$$

We agree to measure the angle θ so that θ = 0 on the boundary OB, and θ = δ on OC. Then the continuity condition for the stress components on the elastic zone boundaries can be written in the form of six equations

$$\begin{gathered}\sigma_r(0) = \sigma_r^-,\quad \sigma_\theta(0) = \sigma_\theta^-,\quad \tau_{r\theta}(0) = \tau_{r\theta}^-,\\ \sigma_r(\delta) = \sigma_r^+,\quad \sigma_\theta(\delta) = \sigma_\theta^+,\quad \tau_{r\theta}(\delta) = \tau_{r\theta}^+.\end{gathered} \tag{9}$$

Four constants are easily determined from the four conjugate conditions for the normal stress components by taking account of (6)-(9)

$$A = -k,\ \ C = -k\cos 2\alpha,\ \ D = -(k+s)/2\delta,\ \ B = (\rho\cos 2\beta + k\cos 2\alpha\cos 2\delta)/\sin 2\delta. \tag{10}$$

Taking account of (6)-(10) we obtain from the two remaining conjugate conditions for the tangential components (9)

$$\begin{gathered}\varphi(\delta) = \rho\cos 2\beta + k\cos 2(\alpha+\delta),\\ \varphi(\delta) = k\cos 2\alpha + \rho\cos 2(\beta+\delta),\end{gathered} \tag{11}$$

where $\varphi(\delta) = (k+s)(2\delta)^{-1}\sin 2\delta$. Equations (11) should be supplemented by the geometric condition

$$\gamma = \alpha + \beta + \delta \tag{12}$$

[α, β, δ, γ are the wedge angular dimensions and three of its zone (Fig. 2)]. The system (11) and (12) completely governs the dependence between the load and the angular dimensions.

Let us make the change of variable

$$2\alpha = \gamma - \delta - \chi,\ 2\beta = \gamma - \delta + \chi, \tag{13}$$

which satisfies condition (12) identically, where $\chi = \beta - \alpha$. Subtracting (11) term by term, we have after manipulation

$$\operatorname{tg}\chi = (k-\rho)(k+\rho)^{-1}\operatorname{tg}\gamma. \tag{14}$$

Combining (11) term by term and taking account of (14), we find

$$\psi^2(\delta) = k^2 + \rho^2 + 2k\rho\cos 2\gamma, \tag{15}$$

where $\psi(\delta) = (k+s)\delta^{-1}\sin\delta$. Since $\rho = s + q$, then (15) governs the dependence of the angular dimension δ of the elastic zone on the applied load q for given parameter k, s, γ. For computations it is more convenient to use the explicit dependence q(δ) obtained from (15):

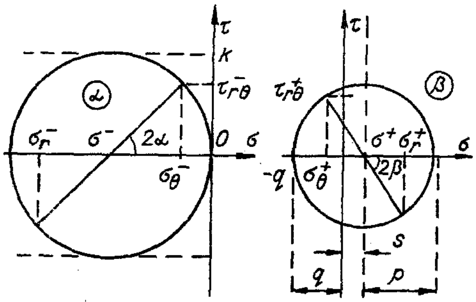

Fig. 3

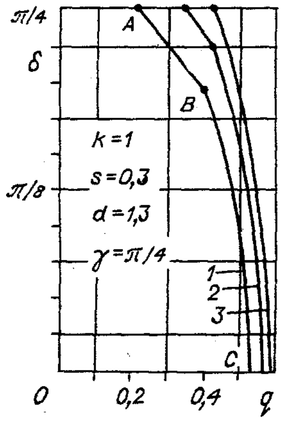

Fig. 4

$$q = -s - k\cos 2\gamma + \sqrt{\psi^2(\delta) - k^2\sin^2 2\gamma}. \tag{16}$$

Evidently, the condition of ultimate resistance to shear occurs in zone β for $\rho = k$. From (15) there thus results

$$q = 2k\,[1 - \delta(\sin\delta)^{-1}\cos\gamma]. \tag{17}$$

This last dependence is presented in [1] for the ultimate resistance of an elastic-plastic wedge just to shear.

For $\delta \to 0$ we have $\psi(\delta) \to k + s$. We obtain from (16) that under the load

$$q_3 = -s - k\cos 2\gamma + \sqrt{(k+s)^2 - k^2\sin^2 2\gamma} \tag{18}$$

the elastic zone δ degenerates into the line separating the zones of ultimate resistance to shear α and ultimate resistance to the mean tensile stress β, where $\alpha + \beta = \gamma$.

Starting from (8) and (10), we determine the difference between the stress components on the boundaries $\theta = 0$, $\theta = \delta$:

$$\begin{aligned}
\tau_{r\theta}(\delta) - \tau_{r\theta}(0) &= (k\cos 2\alpha - \rho\cos 2\beta)\operatorname{tg}\delta,\\
\sigma_\theta(\delta) - \sigma_\theta(0) &= (k+s) - (k\cos 2\alpha + \rho\cos 2\beta),\\
\sigma_r(\delta) - \sigma_r(0) &= (k+s) + (k\cos 2\alpha + \rho\cos 2\beta).
\end{aligned} \tag{19}$$

Therefore, as $\delta \to 0$ the tangential stress component remains continuous since $\tau_{r\theta}(\delta) - \tau_{r\theta}(0) \to 0$. There results from (11) that $k\cos 2\alpha + \rho\cos 2\beta \to k + s$. The normal stress component σ_θ also remains continuous since $\sigma_\theta(\delta) - \sigma_\theta(0) \to 0$. Evidently the component σ_r undergoes a discontinuity:

$$[\sigma_r] = \lim_{\delta\to 0}[\sigma_r(\delta) - \sigma_r(0)] = 2(k+s). \tag{20}$$

Therefore, the elastic zone δ under the load (18) degenerates into a stress discontinuity line. The relationships (19) and (20) and the deductions associated with them hold also for the limit resistance to rupture [4]. The point C, $\delta = 0$, $q = q_3$ corresponds to such a state in the q, δ plane (Fig. 4).

Let us return to (14). According to the sense of the conditions (1)-(3), we have $\rho \leqslant k$. Therefore, $\chi \geqslant 0$, and we establish from (13) that $\alpha \leqslant \beta$, i.e., the dimension of the zone α does not exceed the dimension of the zone β. In the limit case $\alpha \to 0$ we have $\beta + \delta \to \gamma$ from (12). The line $\theta = 0$ becomes the free boundary of the wedge. Taking account of (10) the expressions (8) are reduced to the form

$$\begin{aligned}
&\sigma_r(0),\ \sigma_\theta(0) = (-k) \pm (-k),\\
&\tau_{r\theta}(0) = -(k+s)(2\delta)^{-1} + (\rho\cos 2\beta + k\cos 2\delta)\sin^{-1}2\delta.
\end{aligned} \tag{21}$$

Taking account of the first of the relationships (11), we establish that the conditions

$$\sigma_r = -2k,\ \sigma_\theta = 0,\ \tau_{r\theta} = 0\ (\alpha = 0)$$

are satisfied on the free boundary.

The state mentioned is realized for the load $q = q_2$ which we shall determine below. A further investigation is performed for the parameters having the values

$$\alpha = 0,\ \delta + \beta = \gamma,\ \chi = \beta,\ \sigma_r(0) = \varkappa < k. \tag{22}$$

The inequality $\varkappa < k$ corresponds to the fact that the state of ultimate resistance to shear is not realized on the free boundary of the wedge OA. Only two zones β and δ abutting each other and the faces OD, OA hold in the wedge. Taking account of (22), we find the following relationship from (14)

$$\varkappa = \omega\rho\ (\omega = (\mathrm{tg}\,\gamma + \mathrm{tg}\,\beta)/(\mathrm{tg}\,\gamma - \mathrm{tg}\,\beta),\quad \beta = \gamma - \delta). \tag{23}$$

Taking account of (22) and (23), we obtain a quadratic equation in the parameter ρ from (15)

$$a\rho^2 - 2b\rho - c = 0, \tag{24}$$

where $a = 1 + \omega^2(1 - g^2) + 2\omega\cos 2\gamma$; $b = \omega s g^2$; $c = s^2 g^2$, $g = \delta^{-1}\sin\delta$. Taking into account that $\rho > 0$ in a physical sense, we find from (24)

$$q = \rho - s = (b + \sqrt{b^2 + ac})/a - s. \tag{25}$$

The time of formation of the condition of ultimate resistance to shear on the free boundary corresponds to the characteristic case $\varkappa = k$. Such a state originates for a certain load value $q = q_2$. This value seems to connect the load intervals governed by (16) and (25) for the states of a wedge with the three zones α, β, δ and with the two zones β, δ, respectively. Consequently, to determine the value of q_2 it is necessary to solve (16) and (25) jointly, i.e., to determine the coordinates q_2, δ_2 of the point of intersection B of the curves (16) and (25) on the q, δ plane (see Fig. 4).

Let us consider the limit case $\beta = 0$, $\delta = \gamma$. According to (23), we have $\omega = 1$, $\varkappa = \rho$. The whole wedge is in the elastic state; the state of ultimate resistance to the mean tensile stress $q = q_1$, $\sigma_r(\delta) = q_1 + 2s$ is realized only on the loaded face. The coefficients in (24) take on the values $q = q_1$, $\sigma_r(\delta) = q_1 + 2s$ is realized only on the loaded face. The coefficients in (24) take on the values $a = 4\cos^2\gamma - g^2$, $b = sg^2$, $c = s^2g^2$, $g = \gamma^{-1}\sin\gamma$. From (25) we find the characteristic load

$$q_1 = \rho_1 - s\ (\rho_1 = s(2\gamma\,\mathrm{ctg}\,\gamma - 1)^{-1}), \tag{26}$$

to which the point A in Fig. 4 corresponds. If the mean stress on the loaded face is less than the limit of the resistance to the mean stress ($\sigma < s$), then on this face

$$\rho = \sigma(2\gamma\,\mathrm{ctg}\,\gamma - 1)^{-1} < \rho_1. \tag{27}$$

Therefore, the wedge is completely in the elastic state in the load range $0 < q < q_1$. Two zones hold in the $q_1 \leqslant q < q_2$ range: the elastic δ and the limit β. Three qualitative states of the wedge material are realized for the load $q_2 \leqslant q < q_3$: elastic in the zone δ, limit resistance in the shear zone α and in the mean stress zone β.

The dependence of the angular dimension of the elastic zone that characterizes the wedge carrying capacity, on the applied load $\delta(q)$ is determined by (16) and (25). The example of such a dependence is illustrated by the line 1 in Fig. 4.

Let us note that if the limit of the resistance to the mean stress satisfies the condition $s \geqslant k(2\gamma\cot\gamma - 1)$, then according to (27), the parameter ρ reaches the limit value k of the resistance to shear for $\sigma = k(2\gamma\cot\gamma - 1)$ as q and δ increases, and since $\rho = \varkappa$ in an elastic wedge for $\delta = \gamma$, the conditions of ultimate resistance to shear will occur simultaneously on both faces. As the load increases further these conditions are propagated into the bulk of the wedge. This case is examined in detail in [1]. If $0 < s < k(2\gamma\cot\gamma - 1)$ then according to (26) a limit load q_1 can always be indicated for which the state of ultimate resistance to the mean tensile stress ($s > 0$) will occur. However, for angles satisfying the condition $\gamma\cot\gamma < 1/2$ the expression $2\gamma\cot\gamma - 1 < 0$ results in $0 < s < 0$. This means that the condition of ultimate resistance to the mean tensile stress is unacceptable for a wedge with a sufficiently large aperture angle $\gamma^{-1}\tan\gamma > 2$. Here utilization of the condition of resistance to rupture [4] can be utilized.

An analogous analysis of the ultimate resistance of an elastic-plastic wedge to rupture is given in [3, 4, 10]. Let us note that the relationships (14) and (15) in which it should be taken into account that $s = d - \rho$ can be applied to investigate the ultimate resistance to rupture. From the relationship (15) follows the dependence [4, 10]

$$\delta^{-1}\sin\delta = (k + d - \rho)^{-1}\sqrt{k^2 + \rho^2 + 2k\rho\cos 2\gamma}. \tag{28}$$

Let us convert (28) to the form

$$a\rho^2 + 2b\rho + c = 0, \tag{29}$$

where $a = 1 - g^2$; $b = \lambda g^2 + k \cos 2\gamma$; $c = k^2 - \lambda^2 g^2$; $g = \delta^{-1} \sin \delta$; $\lambda = k + d$. We select the positive value of ρ from the roots of (29) and we determine the dependence between the dimensions of the elastic zone δ and the load q in the interval $q_2 < q < q_3$ for the ultimate resistance to shear and rupture

$$q = 2\rho - d = 2(-b + \sqrt{b^2 - ac})/a - d. \tag{30}$$

In the case of the resistance to rupture for the limit load $q_1 < q < q_2$ we have $\alpha = 0$, $\beta = \gamma - \delta$, $\sigma_r^- = -2\varkappa$ [ϰ is defined exactly as in (23)]. From (15) we obtain

$$a\rho^2 - 2b\rho - c = 0. \tag{31}$$

Here $a = 1 + \omega^2 + 2\omega \cos 2\gamma - (\omega - 1)^2 g^2$; $b = (\omega - 1) d g^2$; $c = d^2 g^2$; $g = \delta^{-1} \sin \delta$.

Determining the positive root ρ from (31), we find the dependence between the load q and the dimension of the elastic zone for the interval $q_1 < q < q_2$ for the ultimate resistance to rupture

$$q = 2\rho - d = 2(b + \sqrt{b^2 + ac})/a - d. \tag{32}$$

The line 2 in Fig. 4 is the function δ(q) given by (30) and (32), for the ultimate resistance of a wedge to shear and rupture, the line 3 is the dependence δ(q) for the ultimate resistance to just shear according to (17). Comparing the graphs shows that taking account of the ultimate resistance to the mean tensile stress can reduce the rated load to a greater degree than taking account of the ultimate resistance to rupture. The characteristic points A, B, C corresponding to the limit loads q_1, q_2, q_3 for which qualitative changes occur in the limit state of an elastic-plastic wedge described above as well as for the ultimate resistance to rupture in [4, 10] hold on curves 1 and 2.

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